

Generalized momenta of mass and their applications to the flow of compressible fluid

Olga Rozanova

Mech.&Math. Faculty, Moscow State University, Moscow, 119992, Russia

Abstract

We present a technique that allows to obtain certain results in the compressible fluid theory: in particular, it is a nonexistence result for the highly decreasing at infinity solutions to the Navier-Stokes equations, the construction of the solutions with uniform deformation and the study of behavior of the boundary of a material volume of liquid.

Key words: compressible fluid, the Cauchy problem, exact solutions, material volume

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1 Main equations

The motion of compressible viscous, heat-conductive, polytropic fluid in $\mathbb{R} \times \mathbb{R}^n$, $n \geq 1$, is governed by the compressible Navier-Stokes (NS) equations

$$\partial_t \rho + \operatorname{div}_x(\rho v) = 0, \quad (1.1)$$

$$\partial_t(\rho v) + \operatorname{div}_x(\rho v \otimes v) + \nabla_x p = \operatorname{Div} T, \quad (1.2)$$

$$\partial_t \left(\frac{1}{2} \rho |v|^2 + \rho e \right) + \operatorname{div}_x \left(\left(\frac{1}{2} \rho |v|^2 + \rho e + p \right) v \right) = \operatorname{div}(T v) + k \Delta_x \theta, \quad (1.3)$$

where ρ , $u = (u_1, \dots, u_n)$, p , e , θ denote the density, velocity, pressure, internal energy and absolute temperature, respectively; T is the stress tensor given by the Newton law $T = T_{ij} = \mu(\partial_i v_j + \partial_j v_i) + \lambda \operatorname{div} u \delta_{ij}$, with constant coefficients of viscosity μ and λ ($\mu \geq 0$, $\lambda + \frac{2}{n}\mu \geq 0$), $k \geq 0$ is the coefficient of heat conduction. We denote Div and div the divergency of tensor and vector,

Email address: rozanova@mech.math.msu.su (Olga Rozanova).

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respectively. The state equations are $p = (\gamma - 1)\rho e$, and $p = R\rho\theta$, where $\gamma > 1$ and $R > 0$ are the specific heat ratio and the universal gas constant, respectively. Thus, we can consider (NS) as a system for unknown ρ , u , p . Indeed, (NS) and the state equations give

$$\partial_t p + (v, \nabla_x p) + \gamma p \operatorname{div} v = (\gamma - 1) \sum_{i,j=1}^n T_{ij} \partial_j v_i + \frac{k}{R} \Delta \frac{p}{\rho}. \quad (1.4)$$

For classical solutions (NS) is equivalent to system (1.1, 1.2, 1.4), denoted (NS*) for short.

(NS*) is supplemented with the initial data

$$(\rho, v, p)|_{t=0} = (\rho_0(x), v_0(x), p_0(x)) \in C^2(\mathbb{R}^n).$$

If $\mu = \lambda = k = 0$, we get the gas dynamic (GD) equations.

2 Conservation laws, generalized momenta of mass and the decay rate

System (NS) is the differential form of conservation laws for the material volume $\mathcal{V}(t)$ (i.e. the volume, consisting from the same particles). (NS) expresses conservation of mass $m = \int_{\mathcal{V}(t)} \rho dx$, momentum $P = \int_{\mathcal{V}(t)} \rho v dx$, and total energy $\mathcal{E} = \int_{\mathcal{V}(t)} \left(\frac{1}{2} \rho |v|^2 + \rho e \right) dx = E_k(t) + E_i(t)$. Here $E_k(t)$ and $E_i(t)$ are the kinetic and internal components of energy, respectively.

When $\mathcal{V}(t) = \mathbb{R}^n$, the conservation of mass, angular momentum and energy take place provided the components of solution decrease at infinity sufficiently quickly.

Definition 2.1 *We say that the classical solution solution to (NS) belongs to the class $K(M(t), \alpha)$, if there exist a positive vector-function $M(t) = (M_v(t), M_\rho(t), M_p(t))$ with components from $C([0, \infty))$, a constant vector $\alpha = (\alpha_v, \alpha_{Dv}, \alpha_\rho, \alpha_p, \alpha_\theta)$ and constants $R_0 > 0$, $T \geq 0$, such that $|v(t, x(t))| \leq M_v(t)|x(t)|^{\alpha_v}$, $|Dv(t, x(t))| \leq M_{Dv}(t)|x(t)|^{\alpha_{Dv}}$, $\rho(t, x(t)) \leq M_\rho(t)|x(t)|^{\alpha_\rho}$, $p(t, x(t)) \leq M_p(t)|x(t)|^{\alpha_p}$, $\theta(t, x(t)) \leq M_\theta(t)|x(t)|^{\alpha_\theta}$ for all trajectories $x(t)$ such that $|x(t)| > R_0$, $t > T$.*

Let us introduce a functional

$$G_\phi(t) = \int_{\mathcal{V}(t)} \rho(t, x) \phi(|x|) dx.$$

If $\phi(|x|) = \frac{1}{2}|x|^2$, then $G_\phi(t)$ is the usual momentum of mass. By analogy for others ϕ we call $G_\phi(t)$ the generalized momentum of mass.

2.1 Decay rate for (NS) equations

Let us choose the appropriate class $K(M(t), \alpha)$ to guarantee conservation of mass, momentum, energy and convergence of the mass momentum $G(t)$ for (NS) system. It is sufficient to set $\alpha = (\alpha_v, \alpha_{Dv}, \alpha_\rho, \alpha_p, \alpha_\theta) = (-n, -n - 1, -n - 2 - \varepsilon, -n - \varepsilon, -n)$ with a constant $\varepsilon > 0$. We denote this class K_{NS} .

If the heat conductivity is zero, we do not need to require the decay of θ , i.e $\alpha = (-n, -n - 1, -n - 2 - \varepsilon, -n - \varepsilon, \alpha_\theta)$ with an arbitrary last component. We denote this class K_{NS_0} .

2.2 Decay rate for (GD) equations

In this case the behavior of velocity is less restrictive. Here it is sufficient to set $\alpha = (\alpha_v, \alpha_{Dv}, -n - 2 - \varepsilon, -n - \varepsilon, \alpha_\theta)$ with $\varepsilon > 0$, $\alpha_v \leq 1$ and arbitrary α_{Dv} and α_θ . The components of velocity may rise as $|\mathbf{x}| \rightarrow \infty$. We denote this class K_{GD} .

3 Behavior of generalized momenta of mass on solutions

We denote $\sigma = (\sigma_1, \dots, \sigma_K)$ the vector with components $\sigma_k = v_i x_j - v_j x_i$, $i > j$, $i, j = 1, \dots, n$, $k = 1, \dots, K$, $K = C_n^2$.

The following lemma describes the behavior of generalized momenta for (GD) system.

Lemma 3.1 *Let us suppose that $\phi(|x|)$ belongs to the class C^2 inside $\mathcal{V}(t)$. For classical solution to system (GD) the following equalities take place:*

$$G'_\phi(t) = \int_{\mathcal{V}(t)} \frac{\phi'(|x|)}{|x|} (v, x) \rho dx,$$

$$G''_\phi(t) = I_{1,\phi}(t) + I_{2,\phi}(t) + I_{3,\phi}(t) + I_{4,\phi}(t),$$

where

$$I_{1,\phi}(t) = \int_{\mathcal{V}(t)} \frac{\phi''(|x|)}{|x|^2} |(v, x)|^2 \rho dx, \quad I_{2,\phi}(t) = \int_{\mathcal{V}(t)} \frac{\phi'(|x|)}{|x|^3} |\sigma|^2 \rho dx,$$

$$I_{3,\phi}(t) = \int_{\mathcal{V}(t)} (\phi''(|x|) + (n-1)\frac{\phi'(|x|)}{|x|})p \, dx, \quad I_{4,\phi}(t) = - \int_{\partial\mathcal{V}(t)} \frac{\phi'(|x|)}{|x|}(x, \nu)p \, d\partial\mathcal{V},$$

where ν is a unit outer normal to \mathcal{V} .

The *proof* is a direct application of the Stokes formula.

Let us consider $\phi(|x|) = \frac{1}{2}|x|^2$ and $\mathcal{V}(t) = \mathbb{R}^n$. We denote the respective functional $G_\phi(t)$ as $G(t)$. In this case the following lemma is true.

Lemma 3.2 *For classical solutions to (NS), $k = 0$, of the class K_{NS_0} and solutions to (GD) of the class K_{GD} the following equality holds:*

$$G''(t) = 2E_k(t) + n(\gamma - 1)E_i(t). \quad (3.1)$$

PROOF. First of all, in (NS) case $G''(t) = I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t)$, where $I_1(t) = \int_{\mathbb{R}^n} \frac{|(u,x)|^2}{|x|^2} \rho \, dx$, $I_2(t) = \int_{\mathbb{R}^n} \frac{|\sigma|^2}{|x|^2} \rho \, dx$, $I_3(t) = n \int_{\mathbb{R}^n} p \, dx$, $I_{4,\phi}(t) = - \lim_{R \rightarrow \infty} \int_{\partial B_R} (x, \nu)p \, d\partial B_R$, $I_5(t) = \int_{\mathbb{R}^n} (\text{Div} T, x) \, dx = \lim_{R \rightarrow \infty} \int_{\partial B_R} T_{ij}x_i\nu_j \, d\partial B_R - (2\mu + n\lambda) \int_{B_R} \text{div} v \, dx = \lim_{R \rightarrow \infty} \int_{\partial B_R} (T_{ij}x_i - (2\mu + n\lambda)v_i \delta_{ij})\nu_j \, d\partial B_R$, where $B_R : |x| \leq R$, we sum the repeated indices. Owing to respective assumptions on the decay of solution as $|\mathbf{x}| \rightarrow \infty$ all improper integrals converge and $I_4(t) = I_5(t) = 0$. To end the proof we note that $I_1(t) + I_2(t) = 2E_k(t)$, $I_3(t) = n(\gamma - 1)E_i(t)$. In (GD) case the integral I_5 is missing.

Remark 3.1 *Equality (3.1) for (GD) was obtained firstly in [1].*

Remark 3.2 *It follows from (3.1) that $n(\gamma - 1)\mathcal{E} \leq G''(t) \leq 2\mathcal{E}$. The inequality implies the same two sided estimate of $G(t)$ that was obtained in 3D for the sum of squared semi-axis of a non-rotating gas ellipsoid next to vacuum [2].*

4 Necessary conditions for existence of global solution

If the initial density ρ_0 is compact, then in arbitrary space dimensions no solution to (NS) from $C^1([0, \infty), H^m(\mathbb{R}^n))$, $m \geq \left[\frac{n}{2}\right] + 2$, exists [3]. This blowup result depends crucially on the assumption about compactness of support of the initial density. Thus, the question remains: is it true that the global in t smooth solution exists for any smooth initial data in the case where the support of initial density coincides with the whole space? Below we find necessary conditions of existence of the global solution with prescribed decay rate as $|x| \rightarrow \infty$.

Theorem 4.1 *If the global in t solution to (NS) of the class K_{NS_0} exists, then the solution components grow as $t \rightarrow \infty$ at least such that*

$$\int_0^t M_v(\tau) d\tau = O(t^{1-\alpha_v}), \quad M_\rho(t) = O(t^{2+\epsilon}), \quad \epsilon > 0, \quad t \rightarrow \infty. \quad (4.1)$$

PROOF. It follows from Lemma 2 that $G''(t) \geq (\gamma - 1)\mathcal{E}$,

$$G(t) \geq \frac{(\gamma - 1)n\mathcal{E}}{2}t^2 + G'(0)t + G(0). \quad (4.2)$$

Let us get the estimate of $G(t)$ from above.

We consider a material volume $\mathcal{V}(t)$ that initially coincides with the ball $|\mathbf{x}| \leq R_0$ (see Definition 1). We denote $B_{R(t)} = (x : |x| \leq R(t))$ a ball that contains $\mathcal{V}(t)$. From the definition of K_{NS} we have

$$\frac{dR(t)}{dt} \leq R^{\alpha_v}(t) M_v(t), \quad R(t) \leq \left((1 - \alpha_v) \int_0^t M_v(\tau) d\tau + R_0^{1-\alpha_v} \right)^{\frac{1}{1-\alpha_v}}.$$

Further,

$$\begin{aligned} G(t) &= \frac{1}{2} \left(\int_{B_{R(t)}} \rho |x|^2 dx + \int_{\mathbb{R}^n \setminus B_{R(t)}} \rho |x|^2 dx \right) \leq \frac{R^2(t)}{2} m + \\ &\quad \frac{1}{2} M_\rho(t) \int_{\mathbb{R}^n \setminus B_{R(t)}} |x|^{2+\alpha_\rho} dx \leq \\ &O \left(\left(\int_0^t M_v(\tau) d\tau \right)^{\frac{2}{1-\alpha_v}} \right) + O \left(M_\rho(t) \left(\int_0^t M_v(\tau) d\tau \right)^{\frac{-\epsilon}{1-\alpha_v}} \right). \end{aligned}$$

If the growth of $M_v(t)$ and $M_\rho(t)$ is less then prescribed in (4.1), then the latter inequality contradicts to (4.2).

Remark 4.1 *In particular, Theorem 1 implies that there exist no smooth solution to (NS) from the class K_{NS_0} with $M_v(t) = \text{const}$ and/or $M_\rho(t) = \text{const}$.*

The analog of Theorem 1 in the case of (GD) equations is the following.

Theorem 4.2 *If the global in t solution to (GD) from the class K_{GD} exists then the functions $M_\rho(t)$ and $M_v(t)$, restricting the density and velocity behavior as $t \rightarrow \infty$, grow at least such that $M_\rho(t) = O(t^{2+\epsilon})$, $\epsilon > 0$, and $\int_0^t M_v(\tau) d\tau = O(\ln t)$ (for $\alpha_v = 1$) or as prescribed in (4.6) (for $\alpha_v < 1$).*

The statement can be proved exactly as Theorem 1.

5 Motion with uniform deformation

We dwell in this section on (GD) equations, however due to the specific choice of the velocity field the results remain true for (NS) with $k = 0$.

The motion with uniform deformation (i.e with linear profile of velocity $v(t, x) = A(t)x$, $A(t)$ is a matrix ($n \times n$)) was considered in many works. Let us mention [4], where this special solution to (GD) was firstly constructed in 3D and [5], where many applications are given. Generally speaking, for this solution the velocity, density and pressure may blow up as $t \rightarrow T < \infty$ and $|\mathbf{x}| \rightarrow \infty$. Below we show how the solution with uniform deformation can be constructed by means of the generalized momenta of mass. Moreover, the requirement of finiteness of energy and mass momentum prohibits the blow up, therefore the respective solution is globally in time smooth.

Below we compare the procedure of constructing of the simplest solution with the linear velocity profile

$$v(t, x) = a(t)x \quad (5.1)$$

for a different choice of generalized momenta.

5.1 Basing on the momentum of mass

Let us consider once more the particular case $\phi(|x|) = \frac{1}{2}|x|^2$, the respective functional is the usual momentum of mass, $G(t)$.

Lemma 2 and (1.4) imply

$$G'(t) = 2a(t)G(t), \quad a'(t) = -a^2(t) + KG^{-\frac{(\gamma-1)n+2}{2}}(t), \quad (5.2)$$

with a constant $K > 0$ depending on initial data. If the initial data satisfy the compatibility condition

$$p'_0(|x|) = -(\gamma - 1)G^{-1}(0)E_i(0)\rho_0(|x|), \quad (5.3)$$

then the density and pressure can be found from (1.1) and (1.4) as

$$\rho(t, |x|) = \exp\left(-n \int_0^t a(\tau)d\tau\right)\rho_0(|x| \exp\left(-\int_0^t a(\tau)d\tau\right)), \quad (5.4)$$

$$p(t, |x|) = \exp(-n\gamma \int_0^t a(\tau) d\tau) \rho_0(|x| \exp(-\int_0^t a(\tau) d\tau)). \quad (5.5)$$

It follows from (5.2) that $a(t)$ solves

$$a'(t) = -a^2(t) + K_1 \exp\left(-((\gamma - 1)n + 2) \int_0^t a(\tau) d\tau\right), \quad (5.6)$$

with $K_1 = K(G(0))^{-(\gamma-1)n+2)/2}$. Since $K_1 > 0$, the solution $a(t)$ remains bounded for all $t \geq 0$ and $a(t) = O(t^{-1})$, $t \rightarrow \infty$ [7].

Remark 5.1 *In the case of globally smooth solution discussed above, the velocity $v(t, x) = a(t)x$, therefore $\alpha_v = 1$. Here $M_v(t) = a(t) = O(t^{-1})$, and $\int_0^t M_v(\tau) d\tau = O(\ln t)$, $M_\rho(t) = O(t^{2+\epsilon})$, $t \rightarrow \infty$. Thus, according Theorem 1, the solution presents a less possible rate of its component in t .*

Remark 5.2 *In [6] the solutions with linear profile of velocity for (GD) equations (including the presence of the Coriolis force and damping) were constructed for a general matrix $A(t)$ in 2D and for particular cases in 3D.*

5.2 The "excluding pressure" case

Let us choose as ϕ a function, proportional to the fundamental solution of the Laplace operator, namely

$$\phi(|x|) = \ln |x|, \quad n = 2, \quad \phi(|x|) = |x|^{2-n}, \quad n \neq 2.$$

In this case Lemma 1 implies

$$G'_\phi(t) = \lambda_1(n) \int_{\mathbb{R}^n} \frac{(v, x)}{|x|^n} \rho dx,$$

$$G''_\phi(t) = \lambda_2(n) \int_{\mathbb{R}^n} \frac{|(v, x)|^2}{|x|^{n+2}} \rho dx + \lambda_3(n) \int_{\mathbb{R}^n} \frac{|\sigma|^2 \rho}{|x|^{n+2}} dx + \frac{p(t, 0)}{\omega_{n-1}(2-n)},$$

where

$$\begin{aligned} \lambda_1(n) &= 1, \quad n = 1, 2 \quad \text{and} \quad 2-n, \quad n \geq 3, \\ \lambda_2(n) &= 0, \quad n = 1; \quad -1, \quad n = 2, \quad \text{and} \quad (1-n)(2-n), \quad n \geq 3, \\ \lambda_3(n) &= 0, \quad n = 1; \quad 1, \quad n = 2, \quad \text{and} \quad 2-n, \quad n \geq 3. \end{aligned}$$

Here we take as $\mathcal{V}(t)$ the space \mathbb{R}^n , all improper integrals are supposed convergent both at the origin and at infinity. Actually, this signifies that $v(t, 0) = 0$.

We use the value of pressure only in the origin; in this sense the pressure in the remaining space is excluded.

Let us consider the case $n \geq 3$. Here $G'_\phi(t) = \lambda_1(n) a(t) G_\phi(t)$,

$$G''_\phi(t) = \lambda_2(n) a^2(t) G_\phi(t) + \frac{p(0,0)}{\omega_{n-1}(2-n)} \exp \left(-\gamma n \int_0^t a(\tau) d\tau \right).$$

The latter system implies

$$a'(t) = -a^2(t) + K_2 \exp \left(-((\gamma - 1)n + 2) \int_0^t a(\tau) d\tau \right), \quad (5.7)$$

with $K_2 = \frac{P(0,0)G_\phi^{n-2}(0)}{\omega_{n-1}(2-n)^2}$. We can see that (5.6) coincides with (5.7), the only difference is in constants K_1 and K_2 . Provided initial data satisfy the compatibility condition

$$G_\phi(0)p'_0(|x|) = -\frac{p(0,0)}{\omega_{n-1}(2-n)^2} \rho_0(|x|)|x|, \quad (5.8)$$

the density and pressure can be found by formulas (5.4), (5.5).

The analysis of equation (5.6) (or (5.7)) shows that if the constant K_1 (or K_2) is positive, as in our case, then the solution exists for all $t \geq 0$ and $\alpha(t) = O(t^{-1})$ as $t \rightarrow \infty$.

We see that in the second case ($\phi = |x|^{2-n}$) we get a more wide class of solutions. Indeed, in Section 5.1 we need to require the significant decay of solutions as $|x| \rightarrow \infty$ to guarantee the finiteness both of energy and momentum of mass, i.e. we restrict ourselves by solutions from K_{GD} with $\alpha_v = 1$. When we construct solutions using the generalized momentum, we do not require the finiteness of energy and the behavior of smooth pressure and density, prescribed by convergence of $G_\phi(t)$ and condition (5.8) is the following:

$$p(|x|) = O(|x|^q), \quad q < 0, \quad \rho(|x|) = O(|x|^s), \quad s = q - 2 < 0, \quad |x| \rightarrow \infty.$$

6 Behavior of boundary of a liquid volume

We mention here very briefly one more application of the generalized momenta of mass to (GD) equations (see [8] for details). Namely, the expansion of boundary of a material volume inside of a smooth flow of gas can be studied by this method. The last question is connected with a problem of air pollution: one can be interested if under usual meteorological conditions a polluted cloud will attain some geographical object.

We suppose that initially a point x_0 do not belong to $\mathcal{V}(t)$. Let us set the following question: what conditions we have to impose on initial data provided

they are known only inside of $\mathcal{V}(0)$, to guarantee that the boundary of given material volume within a smooth flow will attain a given ε - neighborhood of point x_0 ? It is clear that for the answer to this question certain assumptions on the thermodynamic variables in the whole space have to be done. To formulate these assumptions we need the following definition.

Definition 6.1 *We say that the pressure in the moment t is distributed along the boundary $\partial\mathcal{V}(t)$ of domain $\mathcal{V}(t)$, $x_0 \notin \mathcal{V}(t)$, regularly with constant $M \geq 0$, if* $\left| \int_{\partial\mathcal{V}(t)} \left(\frac{x-x_0}{|x-x_0|}, \nu \right) p d\mathcal{V} \right| \leq M$, *where ν is a unit outer normal.*

If $p(t, x)$ is constant, then the integral in the left hand side of the latter inequality is equal to zero. Setting M sufficiently small we assume that the material volume will not meet a zone of large gradient of pressure.

To prove the following theorem we use the generalized momentum of mass with $\phi(|x|) = |x - x_0|^q$, $q < 0$.

Theorem 6.1 [8] *Let a finite material volume $\mathcal{V}(t)$ of compressible liquid with C^1 - smooth boundary $\partial\mathcal{V}(t)$ do not contain a point x_0 . Suppose that the flow is C^1 - smooth for all $t \in [0, T]$, $T \leq \infty$, and the pressure along the boundary $\partial\mathcal{V}(t)$ is distributed regularly with a constant M uniformly in t for $t \in [0, T]$.*

Let us choose some real numbers $q < -n - \frac{2}{\gamma-1}$ and ε , $0 < \varepsilon < \text{dist}(\partial\mathcal{V}(0), x_0)$. Then for all initial data there exists such constant $\delta \leq 0$, depending on initial data, ε , q , T , M , n , γ that if initially

$$\int_{\mathcal{V}(0)} |x - x_0|^{q-2} (v(0, x), x - x_0) \rho_0(x) dx < \delta,$$

then within a time t_1 , later then T , the boundary of given material volume will attain the ε - neighborhood of point x_0 .

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